1.3 LINEAR PROGRAMMING PROBLEM- GRAPHICAL SOLUTION

INTRODUCTION

Graphical method is applicable only if the LPP contains atmost two decision variables.

Feasible Region

A region which satisfies all the constraints including non-negativity constraints is called feasible region.

Graphical Method Procedure1.3.1

*Step*1:

Identify the objective function, decision variables and the constraints.

*Step*2:

Take each of the given constraint as equality sign constraints, since there are either one variable or two variable constraints which represents straight lines, so plot each straight lines of the constraint equation.

*Step*3:

Shade the region of the given constraints which are either below the straight line or above the straight line.

*Step*4:

Find out the feasible region ie common region which satisfies all the constraints.

Step5:

Select the corner point that optimizes (Maximizes or Minimizes) the value of the objective function. It gives the optimum feasible solution.

Note:

(i) If there is no common region, satisfies all the constraints then there are no feasible region in that case given LPP has no feasible solution.

(ii) If the feasible region is unbounded and the objective function is maximize in that case given problem has unbounded solution. If the objective function is minimize we can determine the optimum feasible solution.

(iii) Feasible region occur only in the first quadrant since both the decision variables are ≥ 0.

Example1.3.1

Solve the following LPP using graphical method.

Maximize $Z = 4x_1 + 3x_2$,

subject to the constraints

$$2x_1 + x_2 \leq 1000$$
$$x_1 + x_2 \leq 800$$
$$x_1 \leq 400,\ x_2 \leq 700 \text{ and } x_1, x_2 \geq 0$$

Solution

Determine common region satisfies all the constraints. (Feasible Region).

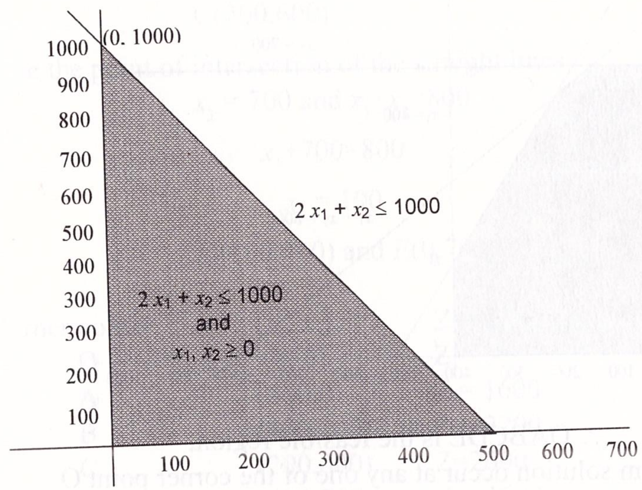
$2x_1 + x_2 = 1000$

Put $x_2 = 0 \therefore 2x_1 = 1000\ x_1 = 500$

$\therefore$ The straight line passes through the point (500, 0)

Put $x_1 = 0 \therefore x_2 = 1000$

$\therefore$ The straight line passes through the point (0, 1000)



But the given constraint is $2x_1 + x_2 \leq 1000$.

$\therefore$ Above shaded region is satisfies the constraint $2x_1 + x_2 \leq 1000$ and $x_1, x_2 \geq 0$.

2[nd] Constraint,

$$x_1 + x_2 = 800$$

Put $x_2 = 0, x_1 = 800$, $\therefore$ the point is (800, 0)

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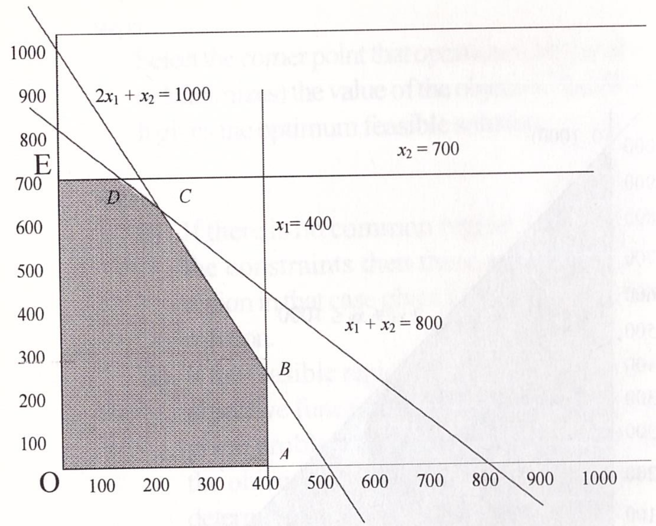

$\therefore$ OABCDE is the feasible region.

Optimum solution occur at any one of the corner point O or A or B or C or D or E.

Corner point O(0,0), A(400,0)

B be the point of intersection of the straight lines $x_1 = 400$ and $2x_1 + x_2 = 1000$

Solving these two equations we get, $x_1 = 400$,

$2(400) + x_2 = 1000$

$x_2 = 200$

$\therefore B(400, 200)$

C be the point of intersection of the straight lines

$2x_1 + x_2 = 1000$ --- (1)

and $x_1 + x_2 = 800$ --- (2)

Solving these two equations

$$\begin{array}{r} 2x_1 + x_2 = 1000 \\ x_1 + x_2 = 800 \\ - \quad - \quad - \\ \hline x_1 \quad = 200 \end{array}$$

Put $x_1 = 200$ in equation 2

$200 + x_2 = 800$

$x_2 = 600$

$\therefore$ C(200,600)

D be the point of intersection of the straight lines

$x_2 = 700$ and $x_1 + x_2 = 800$

$x_1 + 700 = 800$

$x_1 = 100$

$\therefore$D(100,700) and E(0,700)

Corner points	(x_1, x_2)	$Z = 4x_1 + 3x_2$
O	(0, 0)	Z = 0
A	(400,0)	Z = 1600
B	(400,200)	Z = 2200
C	(200,600)	Z=2600
D	(100,700)	Z = 2500
E	(0,700)	Z = 2100

Maximum value occur at the point C (200,600)

$\therefore$ Optimal solution is

$x_1 = 200,\ x_2 = 600$

Maximum of $Z = 2600$

Example1.3.2

Solve the following LPP using graphical method.

Maximize $Z = 3x_1 + 5x_2$, subject to the constraints

$2x_1 + x_2 \geq 7$

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Solve the following LPP using graphical method.

Maximize $Z = 3x_1 + 5x_2$,

subject to the constraints

$2x_1 + x_2 \geq 7$

$x_1 + x_2 \geq 6$

$x_1 + 3x_2 \geq 9$,

$x_1, x_2 \geq 0$

Solution

$2x_1 + x_2 = 7$

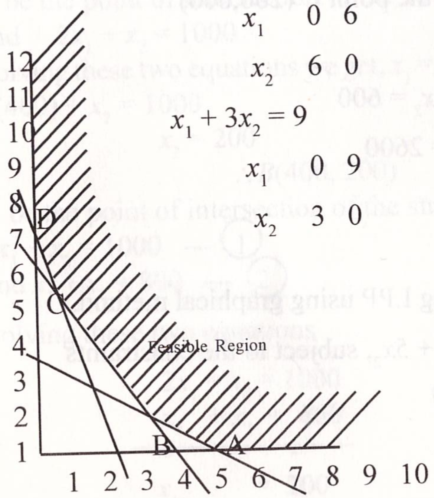
x_1	3.5	0
x_2	0	7

$x_1 + x_2 = 6$

x_1	0	6
x_2	6	0

$x_1 + 3x_2 = 9$

x_1	0	9
x_2	3	0



But the maximum value of the objective function occurs at the point at infinity and hence the problem has an unbounded solution.

Example1.3.3

Solve the following LPP using graphical method.

Minimize $Z = 3x_1 + 5x_2$, subject to the constraints

$2x_1 + x_2 \geq 7$

$x_1 + x_2 \geq 6$

$x_1 + 3x_2 \geq 9$, $x_1, x_2 \geq 0$

Solution

By the previous example feasible region is unbounded. But the minimum value of the objective function occurs at any one of the corner point A, B, C and D.

Corner point	(x_1, x_2)	Min $Z = 3x_1 + 5x_2$
A	(0, 7)	$Z = 35$
B	(1, 5)	$Z = 28$
C	(4.5, 1.5)	$Z = 21$
D	(9, 0)	$Z = 27$

Minimum value of Z occur at the point C

$\therefore$ Optimal solution is $x_1 = 4.5$ and $x_2 = 1.5$

Minimum of $Z = 21$

Current best lower bound. This is the best lower bound (highest in the case of maximization problem and lowest in the case of minimization problem) among the lower bounds of all the fathomed nodes. Initially, it is assumed as infinity for the root node.

Branch-and-bound algorithm applied to maximization problem

Step 1: Solve the given linear programming problem graphically. Set, the current best lower bound, Z_B as ∞.

Step 2: Check, whether the problem has integer solution. If yes, print the current solution as the optimal solution and stop; otherwise go to step 3.

Step 3: Identify the variable X_k which has the maximum fractional part as the branching variable. (In case of tie, select the variable which has the highest objective function coefficient.)

Step 4: Create two more problems by including each of the following constraints to the current problem and solve them.

$$X_k \leq \text{Integer part of } X_k$$

$$X_k \geq \text{Next integer of } X_k$$

Step 5: If any one of the new subproblems has infeasible solution or fully integer values for the decision variables, the corresponding node is fathomed. If a new node has integer values for the decision variables, update the current best lower bound as the lower bound of that node if its lower bound is greater than the previous current best lower bound.

Step 6: Are all terminal nodes fathomed? If the answer is yes, go to step 7; otherwise, identify the node with the highest lower bound and go to step 3.

Step 7: Select the solution of the problem with respect to the fathomed node whose lower bound is equal to the current best lower bound as the optimal solution.

Example 6.8 Solve the following integer programming problem using branch-and-bound technique.

$$\text{Maximize } Z = 10X_1 + 20X_2$$

subject to

$$6X_1 + 8X_2 \leq 48$$

$$X_1 + 3X_2 \leq 12$$

$$X_1, X_2 \geq 0 \text{ and integers}$$

Solution The introduction of the non-negative constraints $X_1 \geq 0$ and $X_2 \geq 0$ will eliminate the second, third and fourth quadrants of the X_1X_2 plane as shown in Figure 6.1.

Now, from the first constraint in equation form

$$6X_1 + 8X_2 = 48$$

we get $X_2 = 6$, when $X_1 = 0$; and $X_1 = 8$, when $X_2 = 0$. Similarly from the second constraint in equation form

$$X_1 + 3X_2 = 12$$

we have $X_2 = 4$, when $X_1 = 0$; and $X_1 = 12$, when $X_2 = 0$.

Now, plot the constraints 1 and 2 as shown in Figure 6.1.

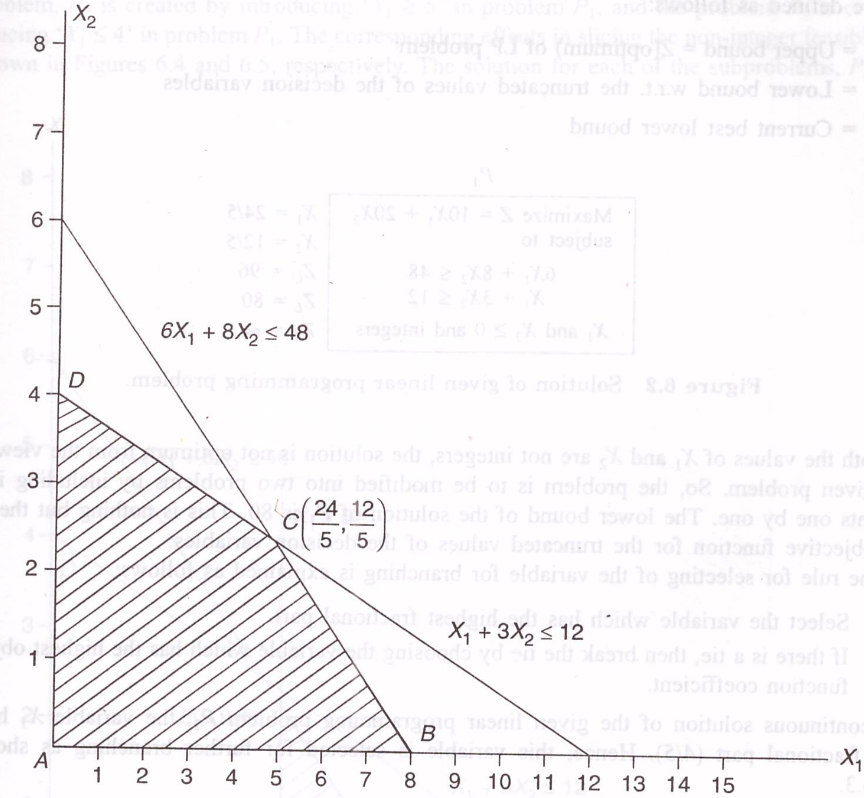


Figure 6.1 Feasible region of Example 6.8.

The closed polygon *ABCD* is the feasible region. The objective function value at each of the corner points of the closed polygon is computed as follows by substituting its coordinates in the objective function:

$$Z(A) = 10 \times 0 + 20 \times 0 = 0$$

$$Z(B) = 10 \times 8 + 20 \times 0 = 80$$

$$Z(C) = 10 \times \frac{24}{5} + 20 \times \frac{12}{5} = 96$$

$$Z(D) = 10 \times 0 + 20 \times 4 = 80$$

Since, the type of the objective function is maximization, the solution corresponding to the maximum Z value is to be selected as the optimum solution. The Z value is maximum for the corner point C. Hence, the corresponding solution of the continuous linear programming problem is presented below.

$$X_1 = \frac{24}{5},\ X_2 = \frac{12}{5},\ Z(\text{optimum}) = 96$$

These are jointly shown as problem P_1 in Figure 6.2. The notations for different types of lower bound are defined as follows:

Z_U = Upper bound = Z(optimum) of LP problem

Z_L = Lower bound w.r.t. the truncated values of the decision variables

Z_B = Current best lower bound

P_1

Maximize $Z = 10X_1 + 20X_2$ subject to $6X_1 + 8X_2 \le 48$ $X_1 + 3X_2 \le 12$ X_1 and $X_2 \ge 0$ and integers	$X_1 = 24/5$ $X_2 = 12/5$ $Z_U = 96$ $Z_L = 80$ $Z_B = \infty$

Figure 6.2 Solution of given linear programming problem.

Since both the values of X_1 and X_2 are not integers, the solution is not optimum from the view point of the given problem. So, the problem is to be modified into two problems by including integer constraints one by one. The lower bound of the solution of P_1 is 80. This is nothing but the value of the objective function for the truncated values of the decision variables.

The rule for selecting of the variable for branching is explained as follows:

1. Select the variable which has the highest fractional part.
2. If there is a tie, then break the tie by choosing the variable which has the highest objective function coefficient.

In the continuous solution of the given linear programming problem P_1, the variable X_1 has the highest fractional part (4/5). Hence, this variable is selected for further branching as shown in Figure 6.3.

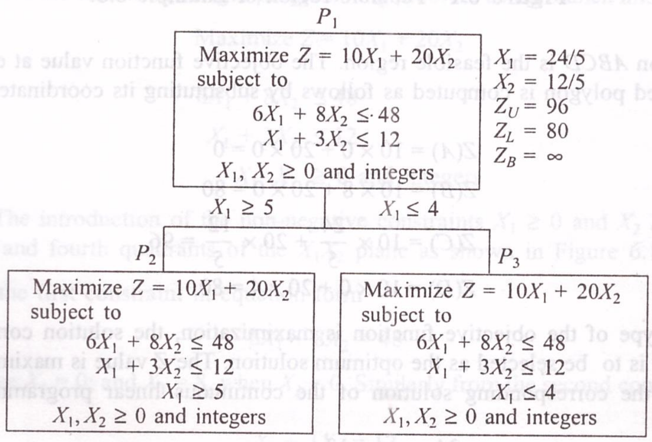


Figure 6.3 Branching from P_1.

In Figure 6.3, the problems, P_2 and P_3 are generated by adding an additional constraint. The subproblem, P_2 is created by introducing '$X_1 \geq 5$' in problem P_1, and the problem P_3 is created by introducing '$X_1 \leq 4$' in problem P_1. The corresponding effects in slicing the non-integer feasible region are shown in Figures 6.4 and 6.5, respectively. The solution for each of the subproblems, P_2 and P_3

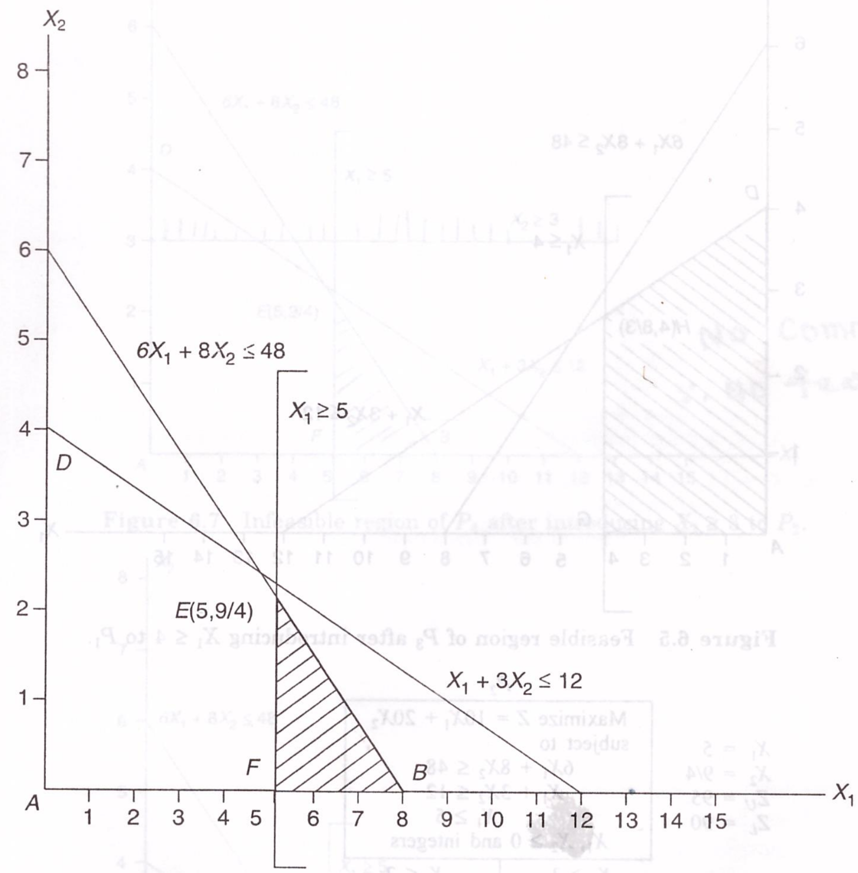


Figure 6.4 Feasible region of P_2 after introducing $X_1 \geq 5$ to P_1.

is obtained from Figures 6.4 and 6.5, respectively. These are summarized in Figure 6.3. The problem P_2 has the highest lower bound of 90 among the unfathomed terminal nodes. So, the further branching is done from this node as shown in Figure 6.6.

In Figure 6.6, the problems, P_4 and P_5 are generated by adding an additional constraint to P_2. The problem, P_4 is created by including '$X_2 \geq 3$' in problem P_2, and problem P_5 is created by including '$X_2 \leq 2$' in problem P_2. The corresponding effects in slicing the non-integer feasible region are shown in Figures 6.7 and 6.8, respectively. The solution for each of the problems P_4 and P_5 is obtained from Figures 6.7 and 6.8, respectively. The problem P_4 has infeasible solution. So, this node is fathomed. The lower bound of the node P_5 is 90. But, the solution of the node P_5 is still non-integer. Now, the

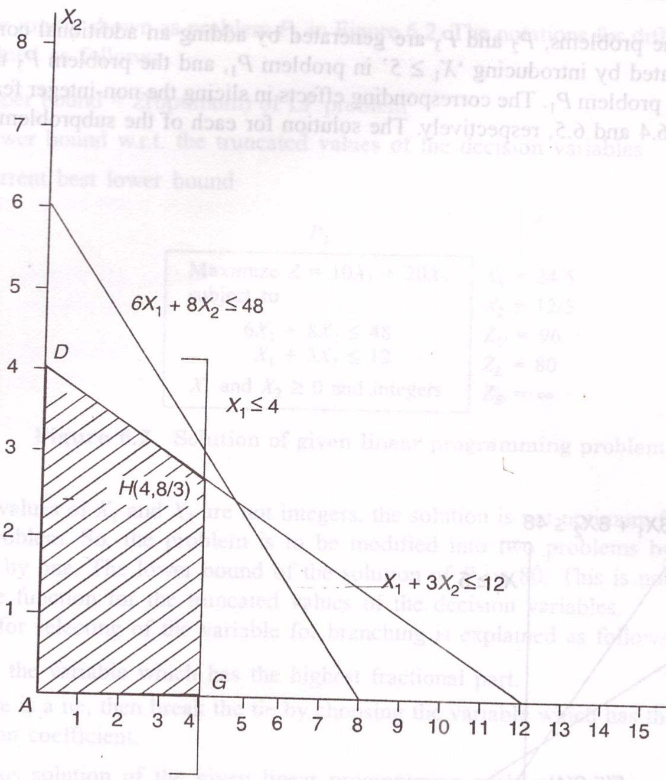


Figure 6.5 Feasible region of P_3 after introducing $X_1 \le 4$ to P_1.

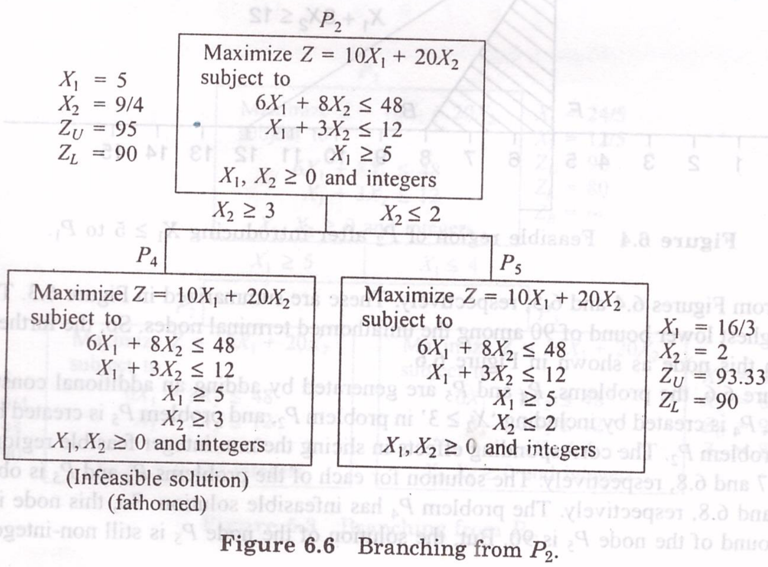


Figure 6.6 Branching from P_2.

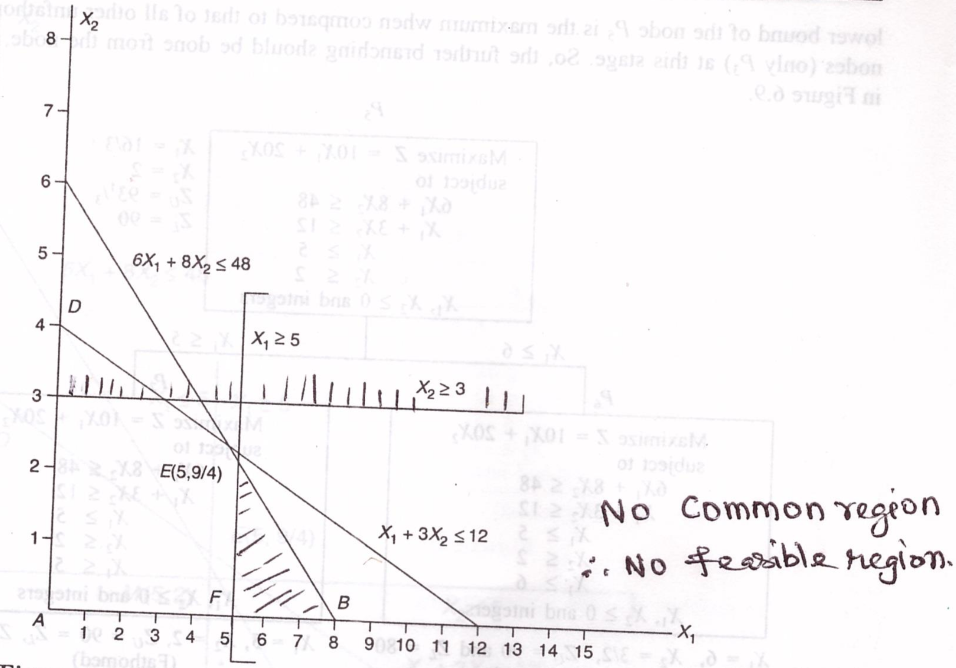


Figure 6.7 Infeasible region of P_4 after introducing $X_2 \ge 3$ to P_2.

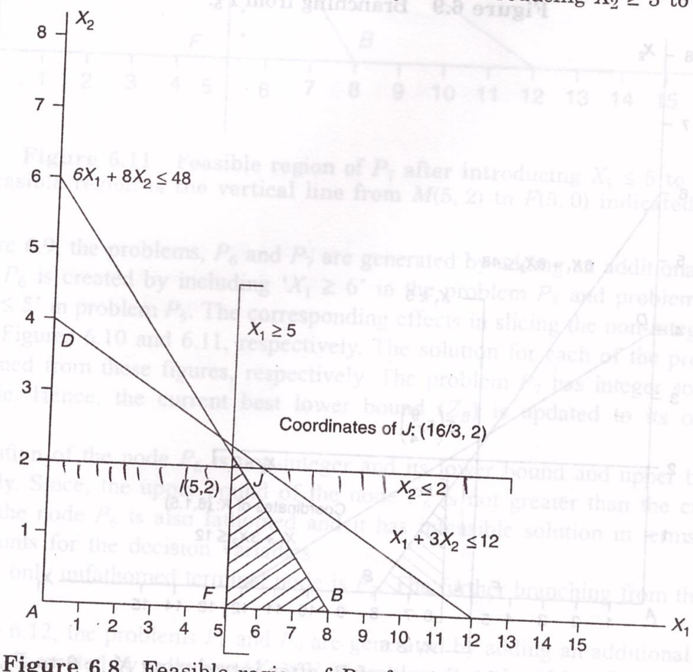


Figure 6.8 Feasible region of P_5 after introducing $X_2 \le 2$ to P_2.

lower bound of the node P_5 is the maximum when compared to that of all other unfathomed terminal nodes (only P_3) at this stage. So, the further branching should be done from the node, P_5 as shown in Figure 6.9.

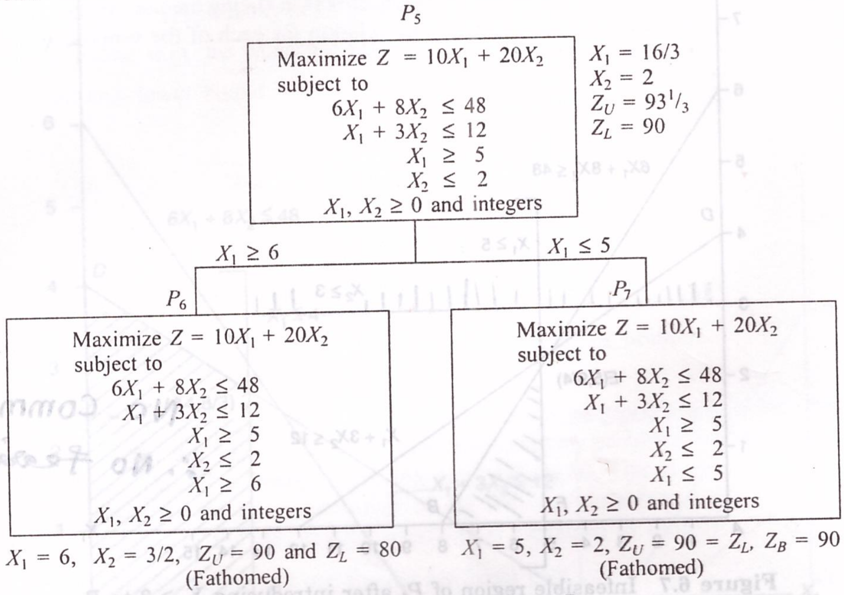


Figure 6.9 Branching from P_5.

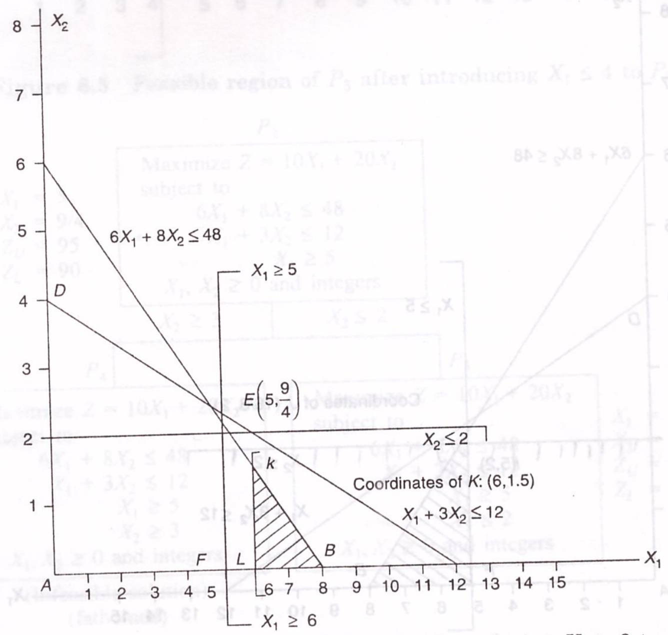


Figure 6.10 Feasible region of P_6 after introducing $X_1 \geq 6$ to P_5.

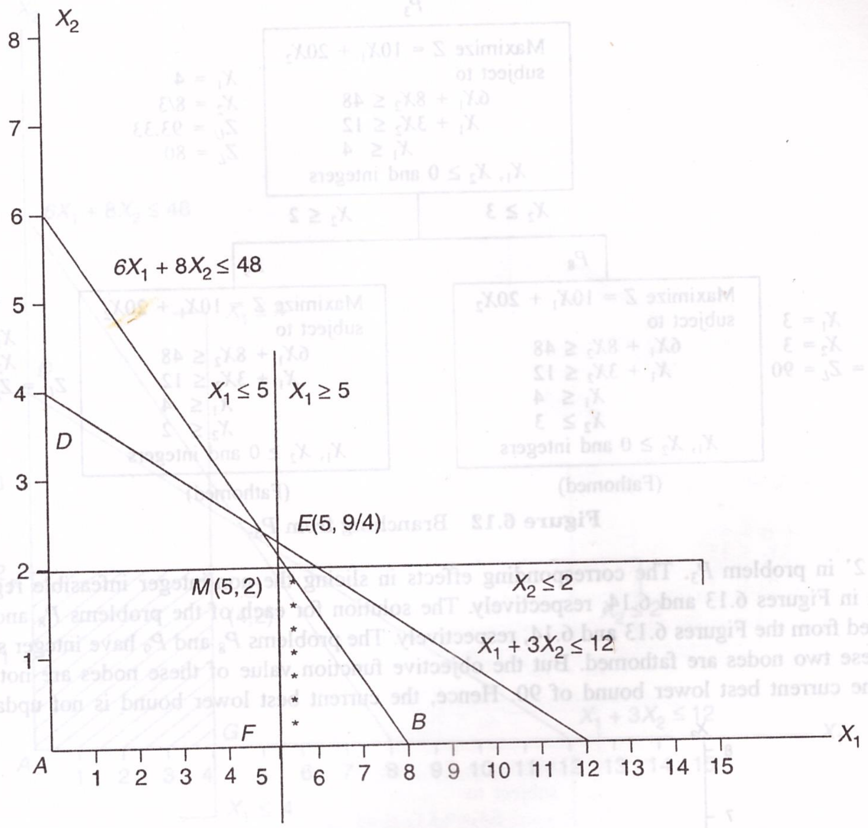


Figure 6.11 Feasible region of P_7 after introducing $X_1 \le 5$ to P_5.
[Feasible region is the vertical line from $M(5, 2)$ to $F(5, 0)$ indicated by *s.]

In Figure 6.9, the problems, P_6 and P_7 are generated by adding an additional constraint to P_5. The problem P_6 is created by including '$X_1 \ge 6$' in the problem P_5 and problem P_7 is created by including '$X_1 \le 5$' in problem P_5. The corresponding effects in slicing the non-integer feasible region are shown in Figures 6.10 and 6.11, respectively. The solution for each of the problems P_6 and P_7 are also obtained from these figures, respectively. The problem P_7 has integer solution. So, it is a fathomed node. Hence, the current best lower bound (Z_B) is updated to its objective function value, 90.

The solution of the node P_6 is non-integer and its lower bound and upper bound are 80 and 90, respectively. Since, the upper bound of the node P_6 is not greater than the current best lower bound of 90, the node P_6 is also fathomed and it has infeasible solution in terms of not fulfilling integer constraints for the decision variables.

Now, the only unfathomed terminal node is P_3. The further branching from this node is shown in Figure 6.12.

In Figure 6.12, the problems P_8 and P_9 are generated by adding an additional constraint to P_3. The problem P_8 is created by including '$X_2 \ge 3$' in problem P_3 and problem P_9 is created by including

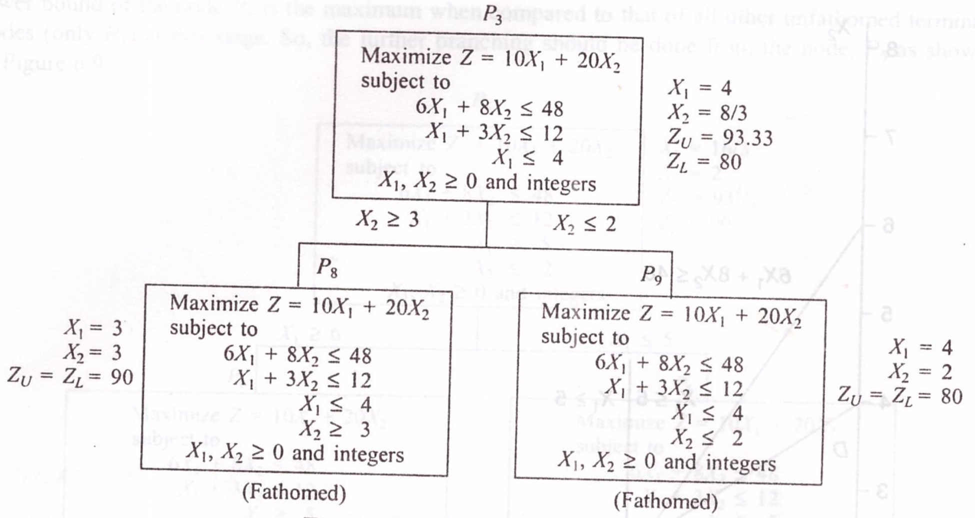


Figure 6.12 Branching from P_3.

'$X_2 \le 2$' in problem P_3. The corresponding effects in slicing the non-integer infeasible region are shown in Figures 6.13 and 6.14, respectively. The solution for each of the problems P_8 and P_9 are obtained from the Figures 6.13 and 6.14, respectively. The problems P_8 and P_9 have integer solution. So, these two nodes are fathomed. But the objective function value of these nodes are not greater than the current best lower bound of 90. Hence, the current best lower bound is not updated.

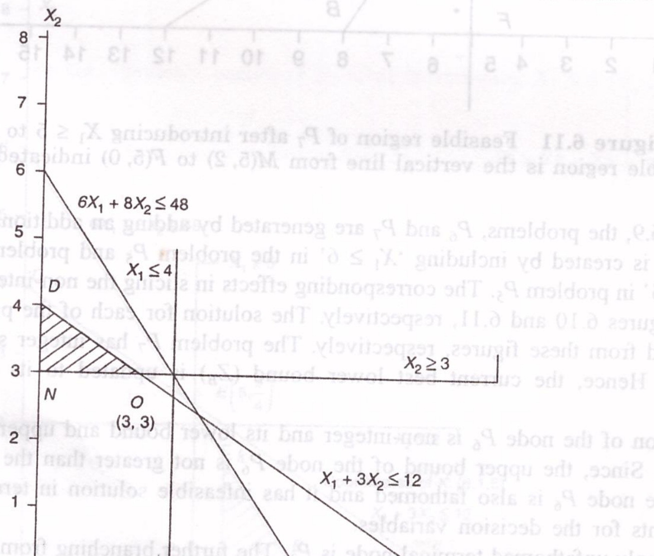


Figure 6.13 Feasible region of P_8 after introducing $X_2 \ge 3$ to P_3.

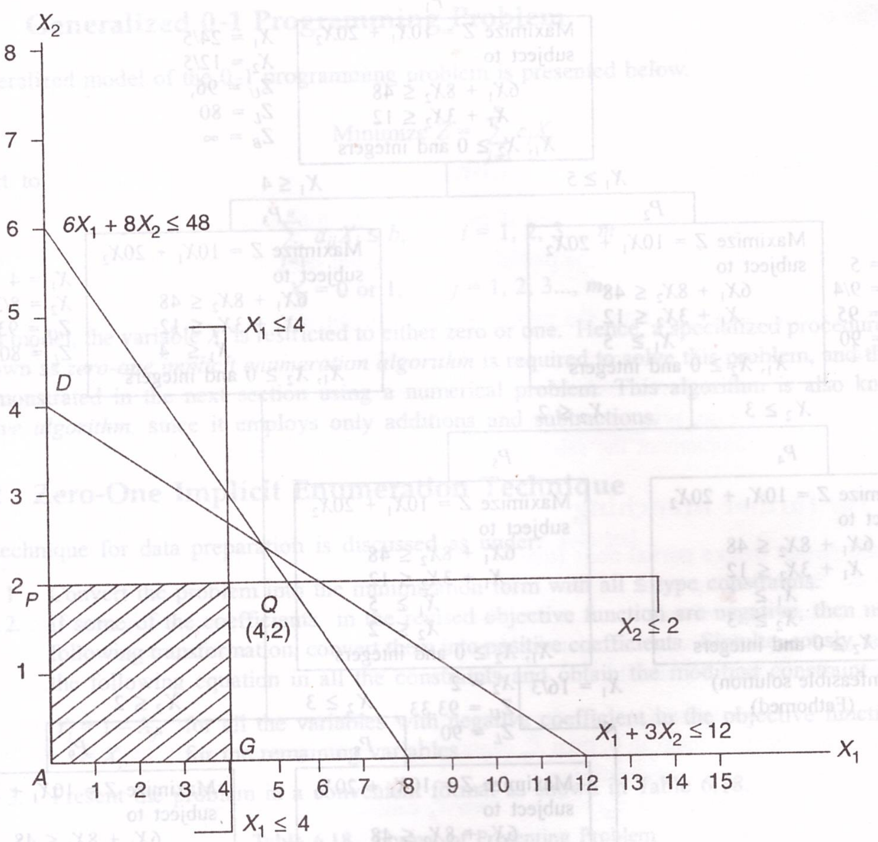


Figure 6.14 Feasible region of P_9 after introducing $X_2 \le 2$ to P_3.

Now, all the terminal nodes are fathomed. The feasible fathomed node with the current best lower bound is P_7. Hence, its solution is treated as the optimal solution as listed below. A complete branching tree is shown in Figure 6.15.

$$X_1 = 5, \quad X_2 = 2, \quad Z(\text{optimum}) = 90$$

Note: This problem has alternate optimum solution at P_8 with $X_1 = 3$, $X_2 = 3$, Z(optimum) = 90.

6.5 ZERO-ONE IMPLICIT ENUMERATION ALGORITHM

Zero-one (0-1) programming is a special kind of linear programming problem. In this type of problem, all the variables are restricted to either 0 or 1. This type of problem exists in many realistic situations like, capital budgeting problem, assignment problem, scheduling problem, portfolio problem, etc.

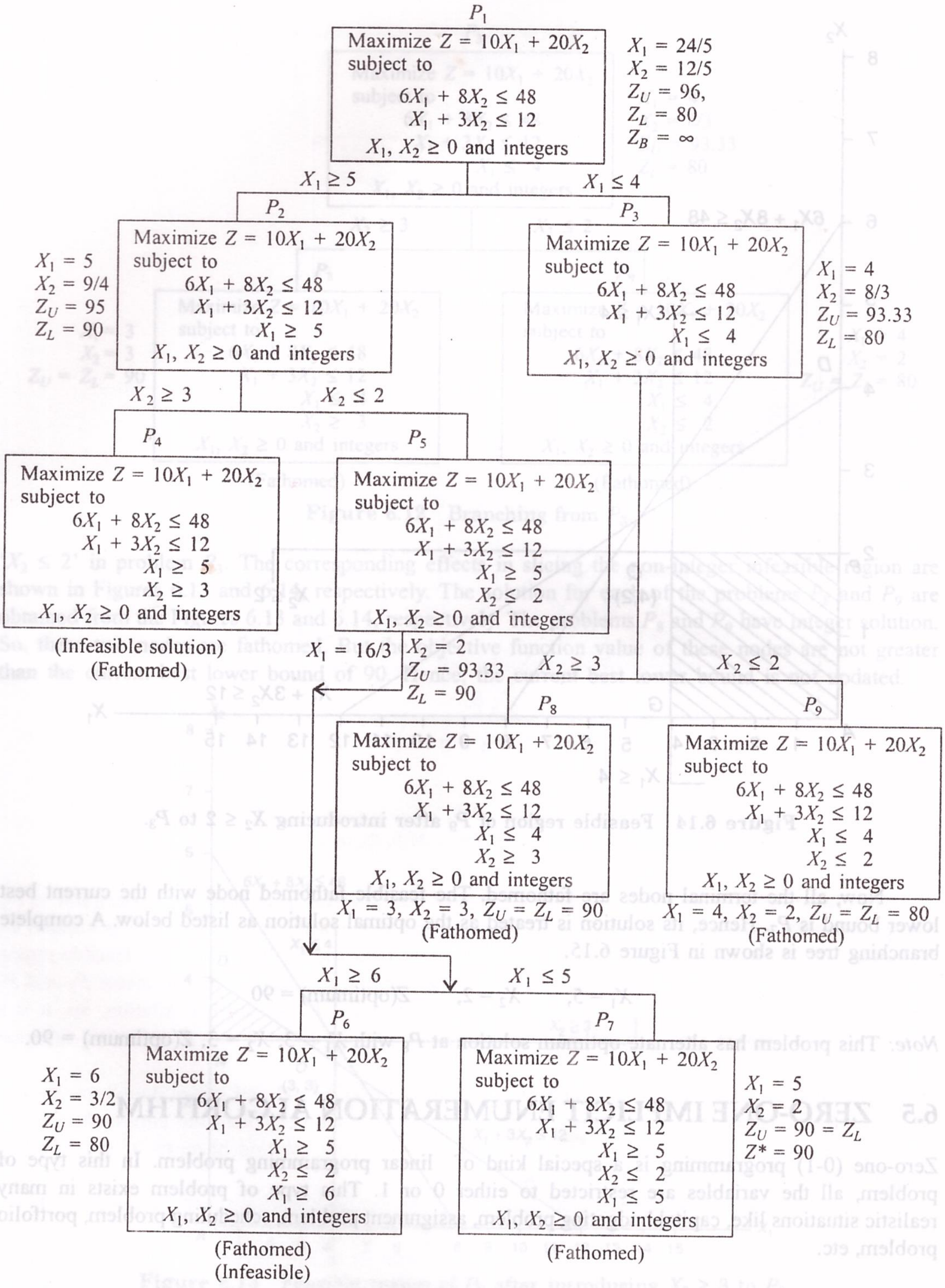


Figure 6.15 Complete tree of Example 6.8.